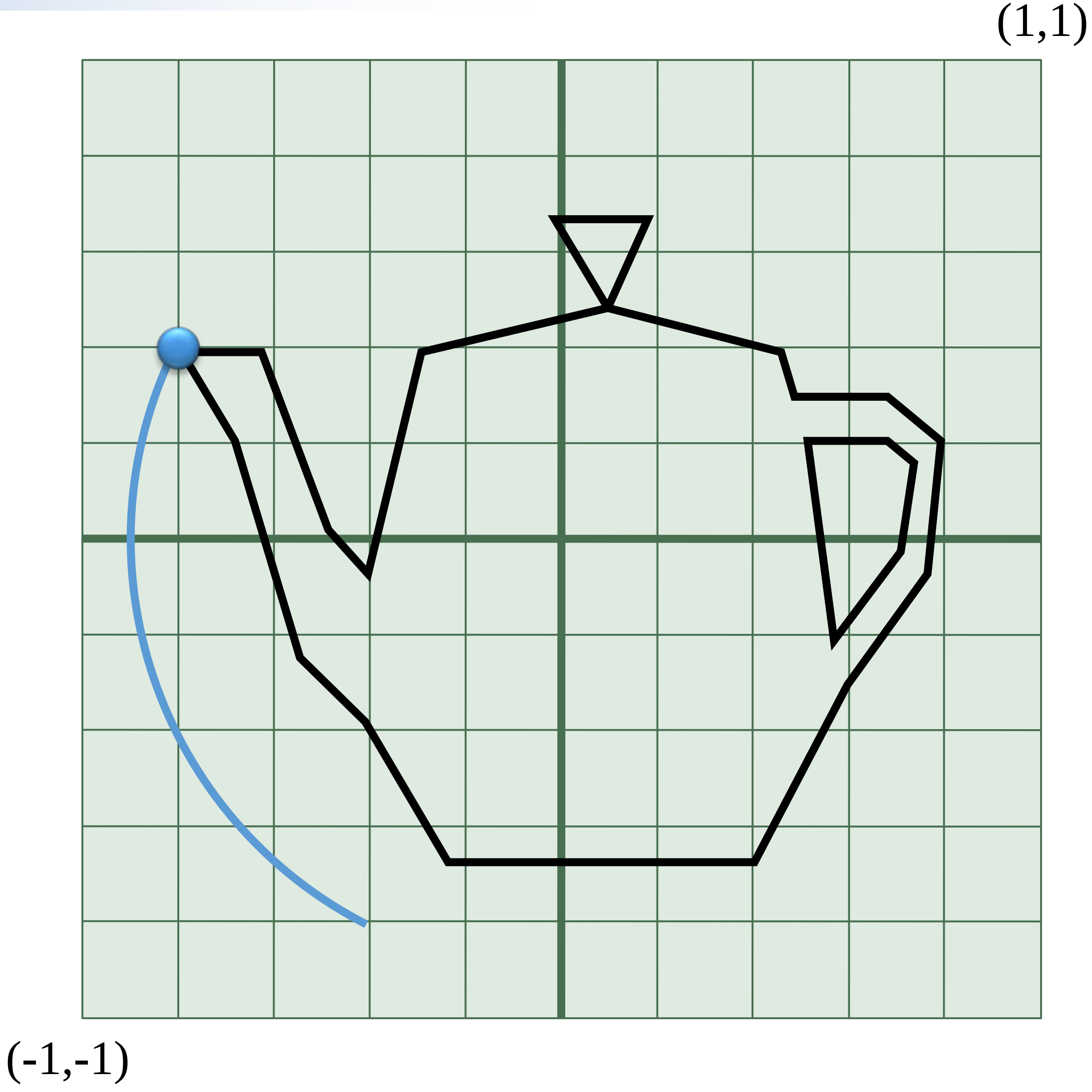
Transformational Geometry

CS418 Computer Graphics

John C. Hart

2-D Points

- The position of the 2-D point (x,y) is represented in the plane by the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$, for example $\begin{bmatrix} -0.8 \\ 0.4 \end{bmatrix}$
- Represent a 2-D shape with polygons connecting vertices at 2-D positions
- Transform an object by transforming the positions of its vertices



2-D Points

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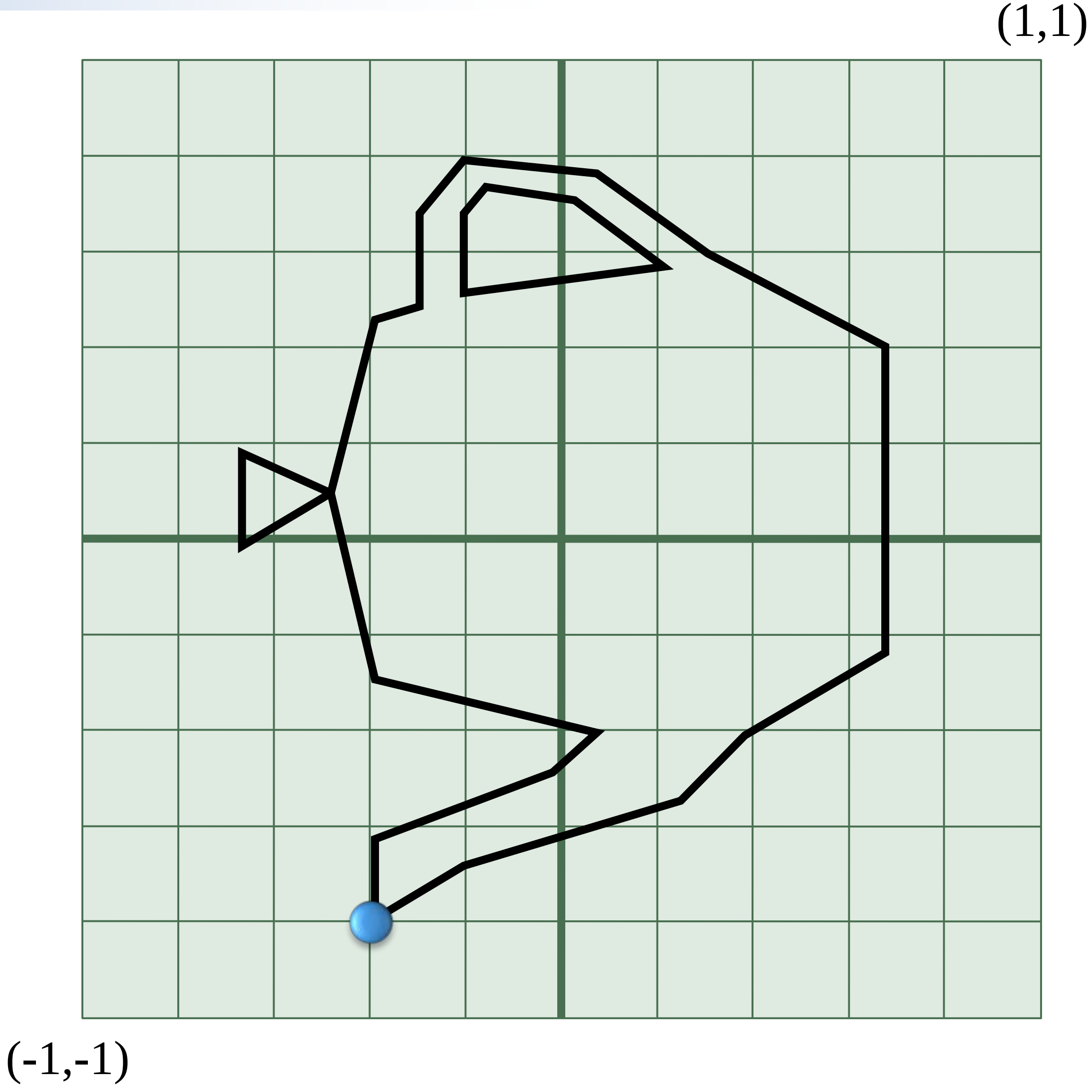
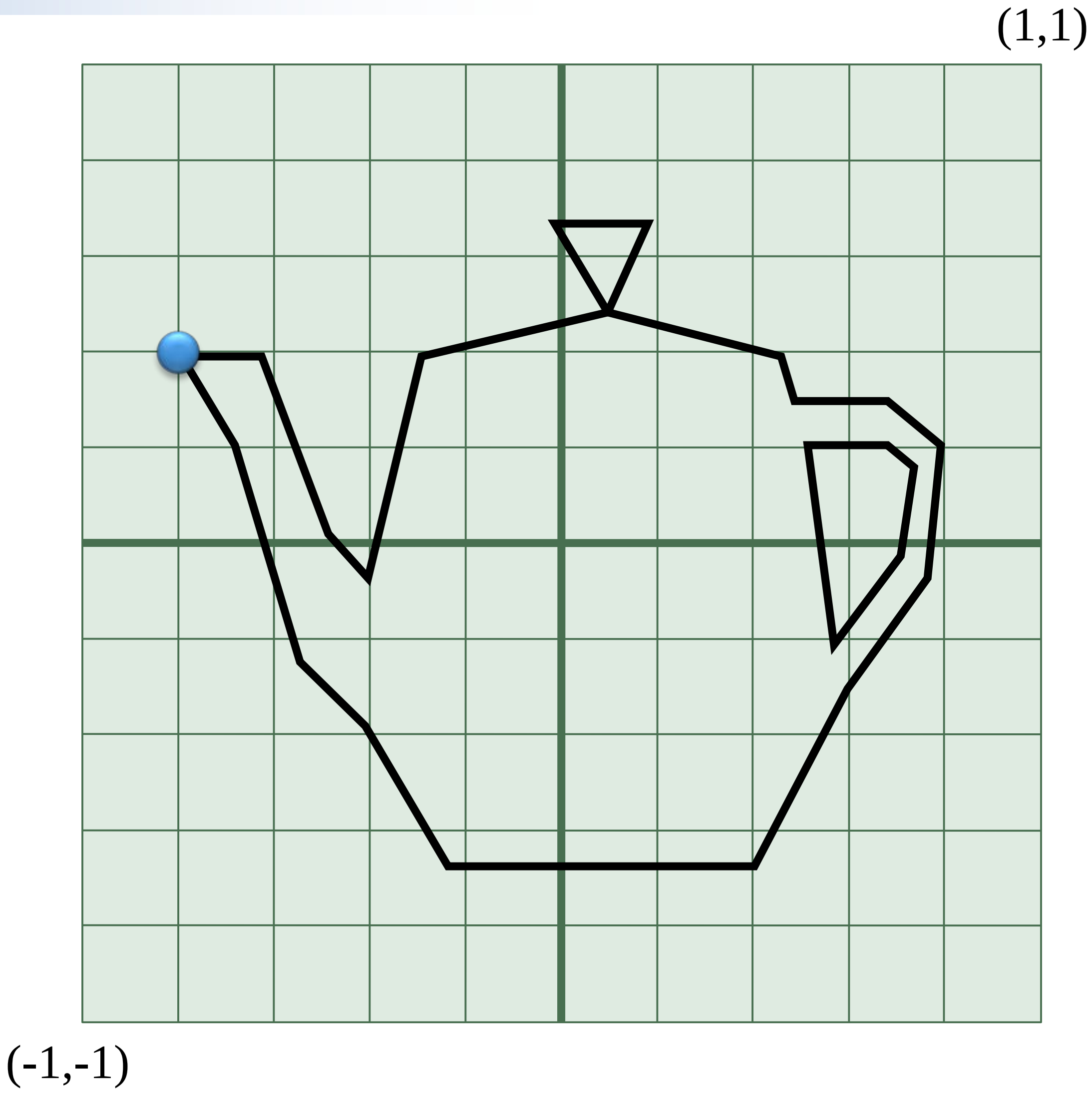
- Represent a 2-D shape with polygons connecting vertices at 2-D positions
- Transform an object by transforming the positions of its vertices,

for example $\begin{bmatrix} -0.4 \\ -0.8 \end{bmatrix}$



2-D Transformations

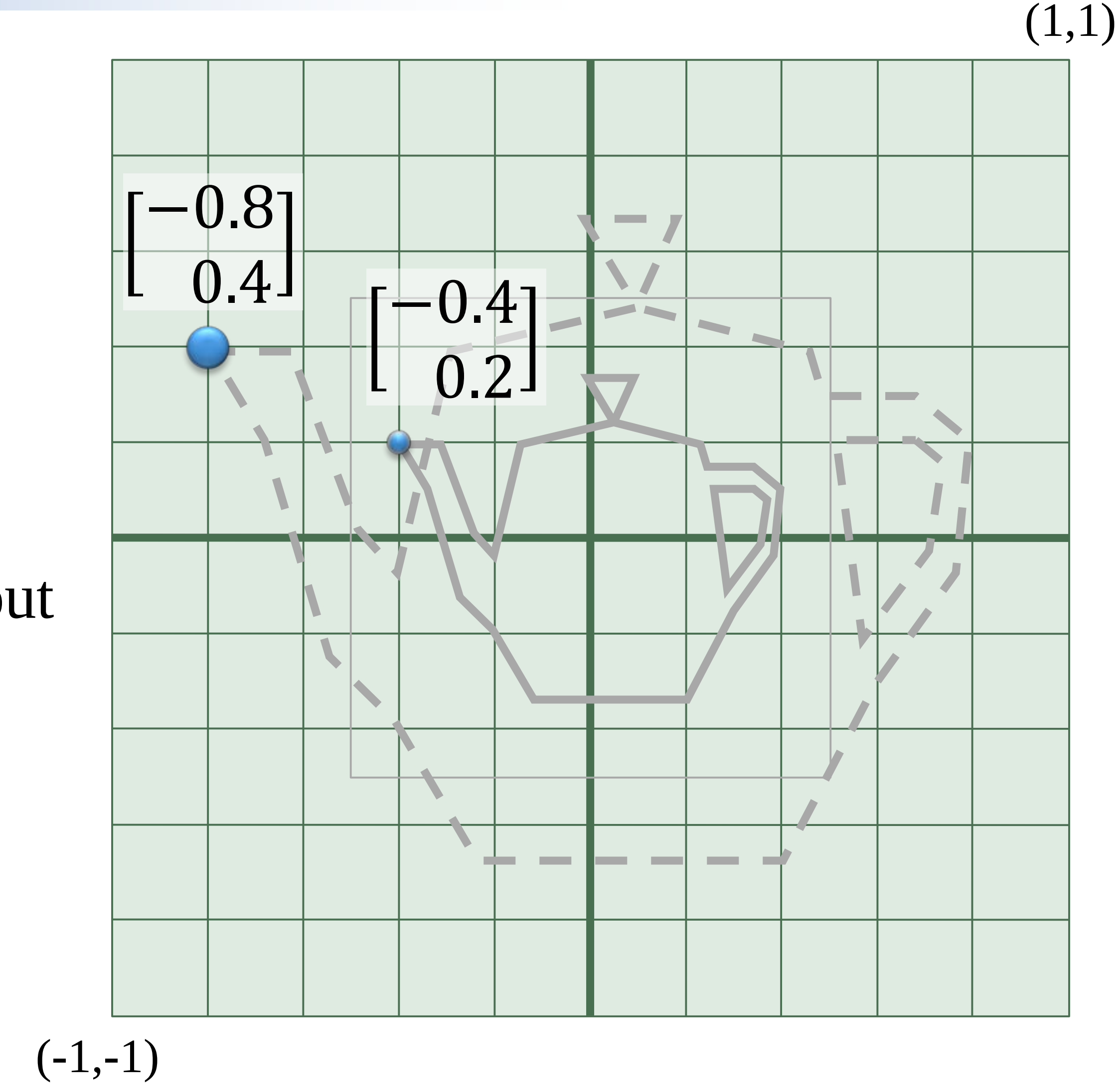
- Scale
- Translate
- Rotate



Scale

- Scale to ½ size
- Multiply vertex position coordinates by 0.5
- Scale relative to origin
  - Shrinking pulls vertices in closer to the origin
  - Expanding pushes vertices out farther away from the origin

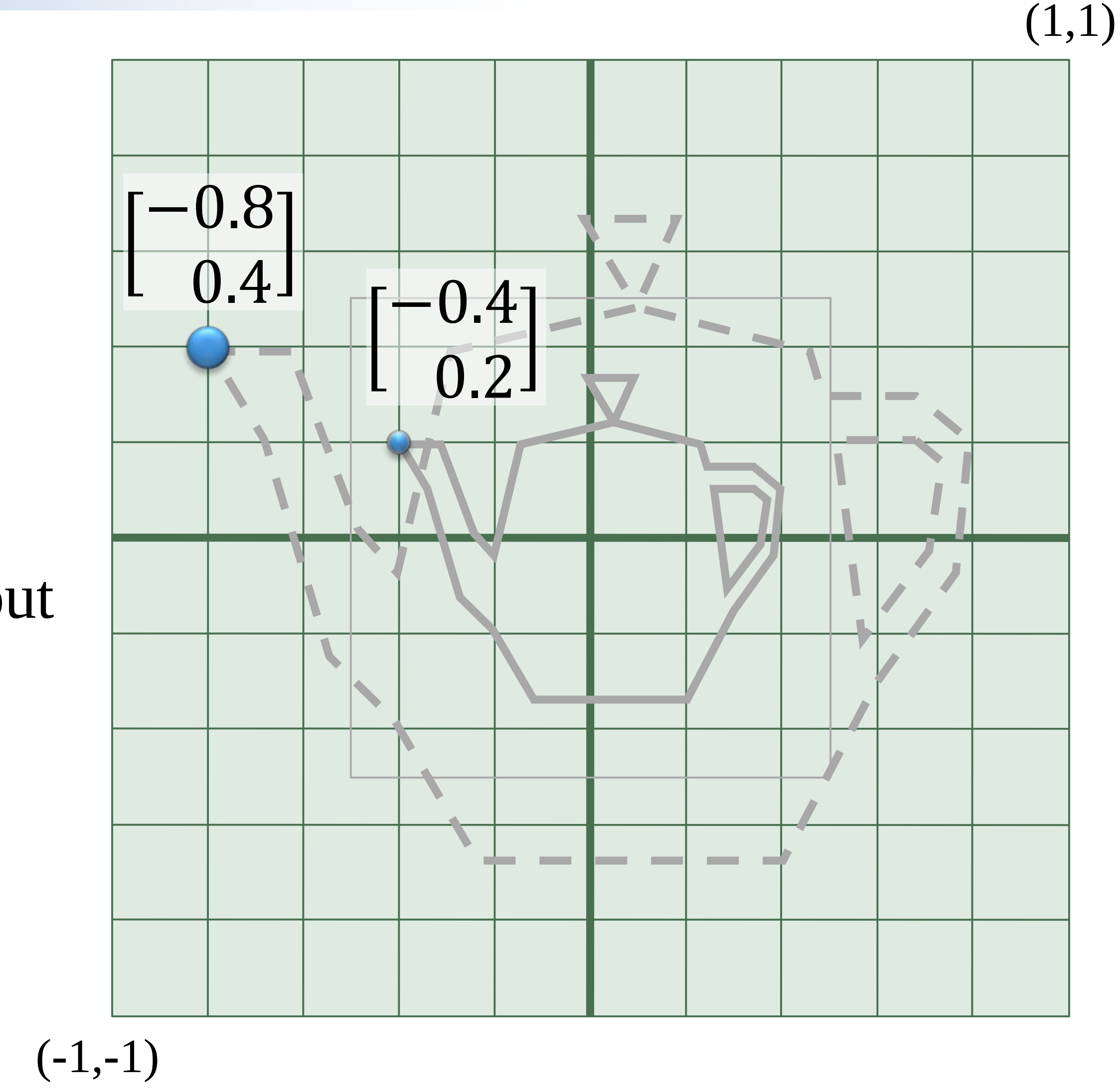
$$0.5 \begin{bmatrix} -0.8 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.2 \end{bmatrix}$$



Scale

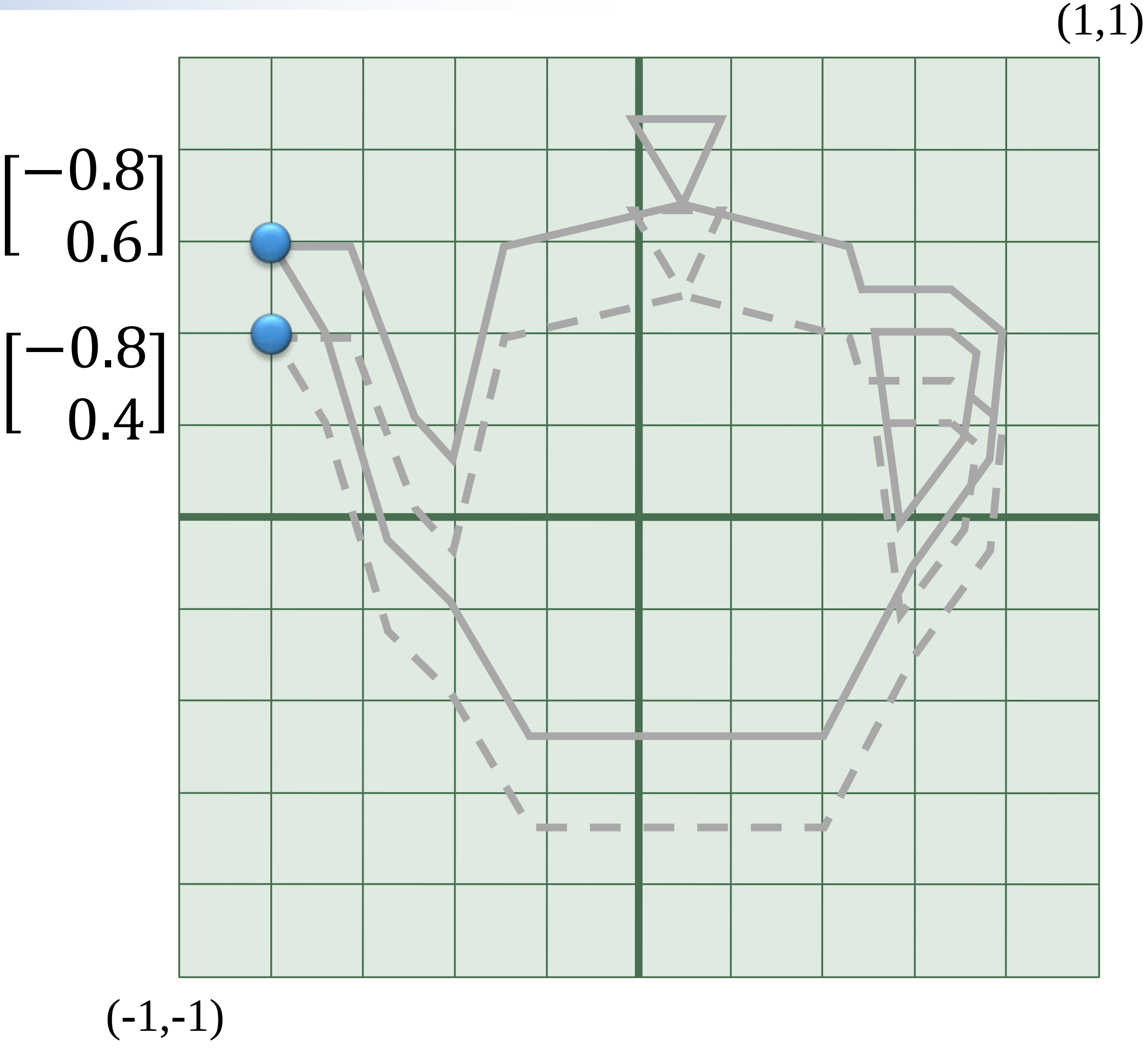
- Scale to ½ size
- Multiply vertex position coordinates by 0.5
- Scale relative to origin
  - Shrinking pulls vertices in closer to the origin
  - Expanding pushes vertices out farther away from the origin

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.2 \end{bmatrix}$$



Translate

- Translate up by 0.2
- Add 0.2 to the y-coordinate of the position of the vertices



$$\begin{bmatrix}-0.8\\0.4\end{bmatrix}+\begin{bmatrix}0.0\\0.2\end{bmatrix}=\begin{bmatrix}-0.8\\0.6\end{bmatrix}$$

Homogeneous Coordinates

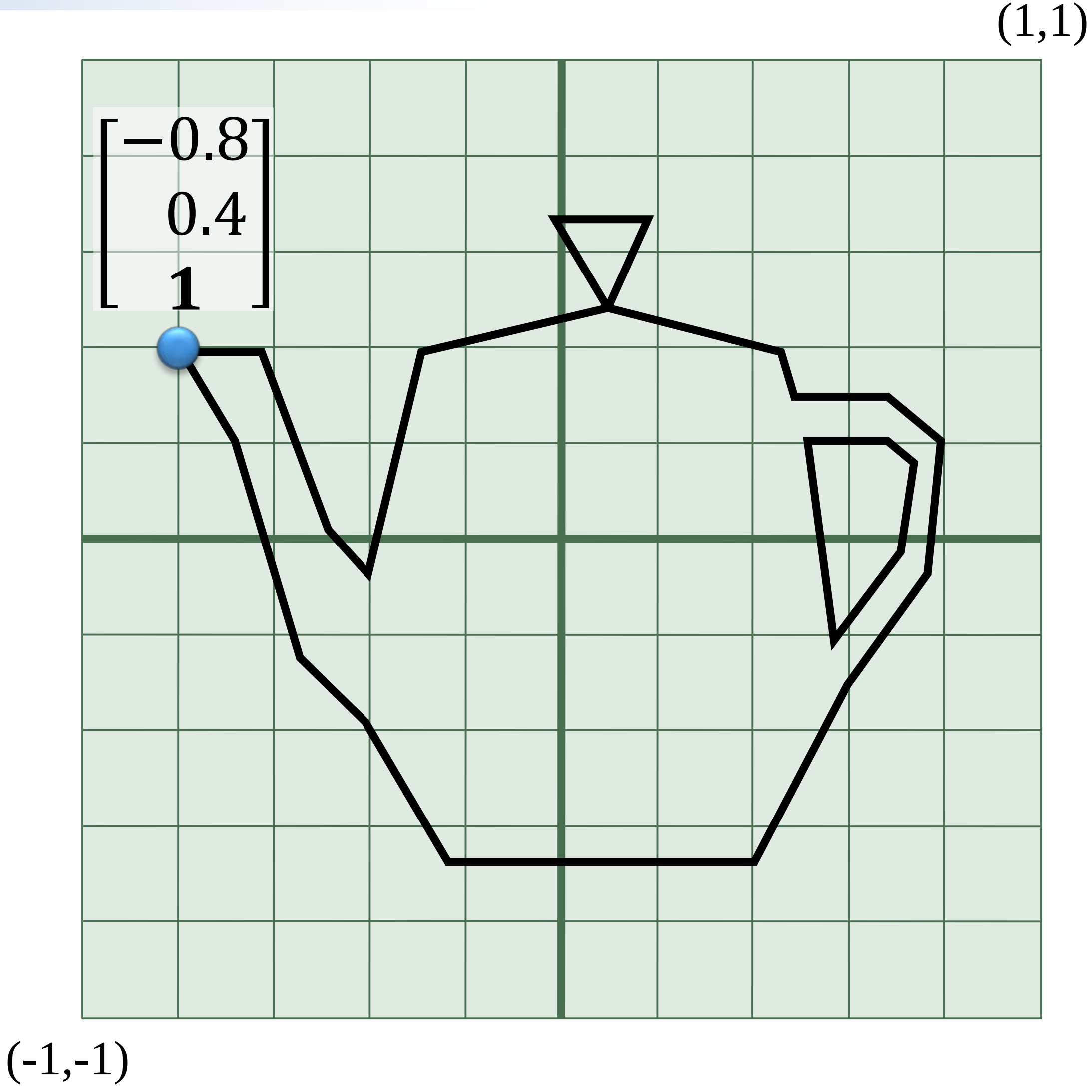
$$S\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix},\ S\left(\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}\right)$$

- Irregular to handle scale differently than translate
- Need to implement translation using matrix multiplication

$$TS\begin{bmatrix}x\\y\end{bmatrix},\ ST\begin{bmatrix}x\\y\end{bmatrix}$$

Solution: add a third "homogenous coordinate" to the vertex position column vector

$$\begin{bmatrix}x\\y\\\mathbf{1}\end{bmatrix}$$



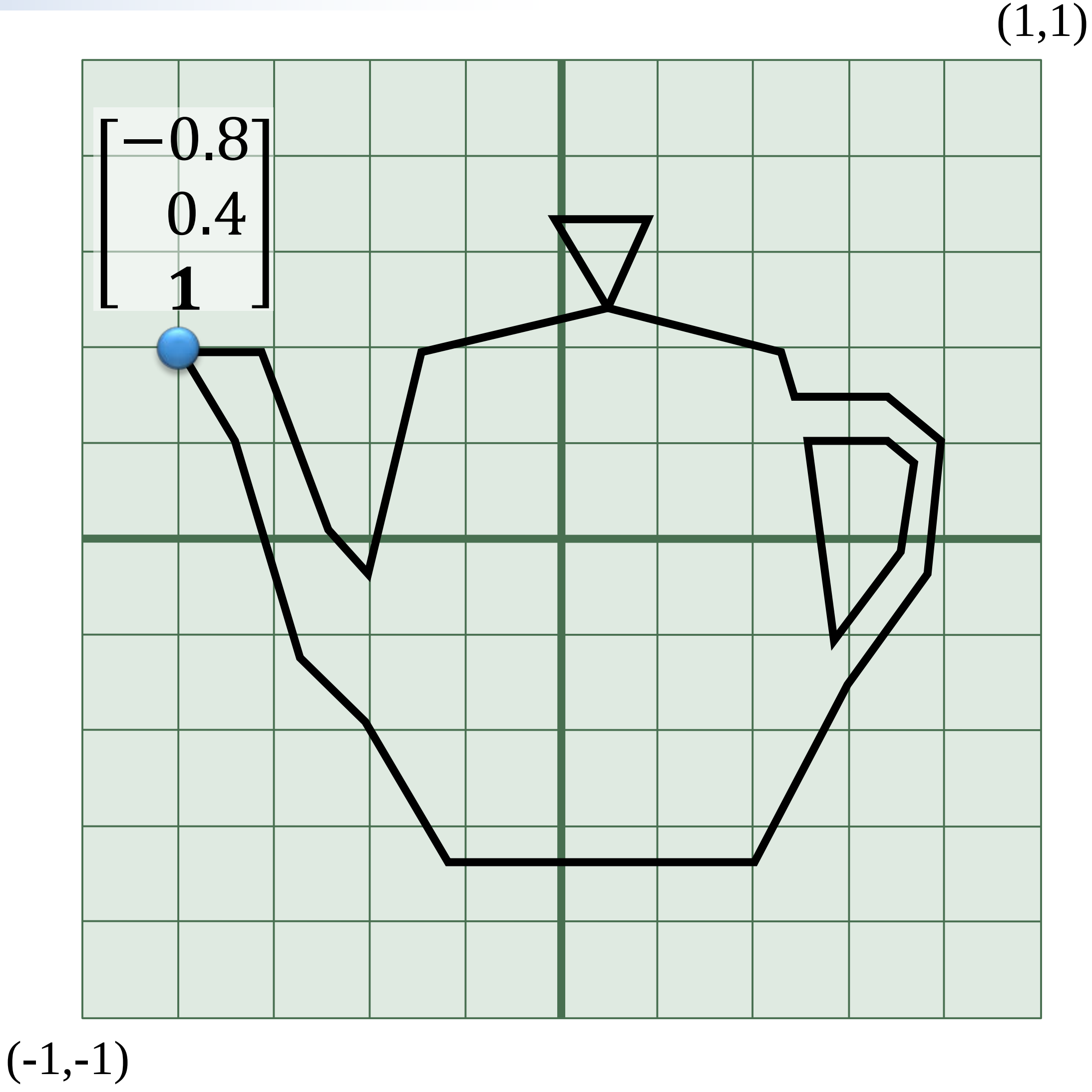
Homogeneous Coordinates

- Scale

$$\begin{bmatrix} s & & \\ & s & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} sx \\ sy \\ 1 \end{bmatrix}$$

- Translate

$$\begin{bmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$



Transformation Composition

- Scale then translate

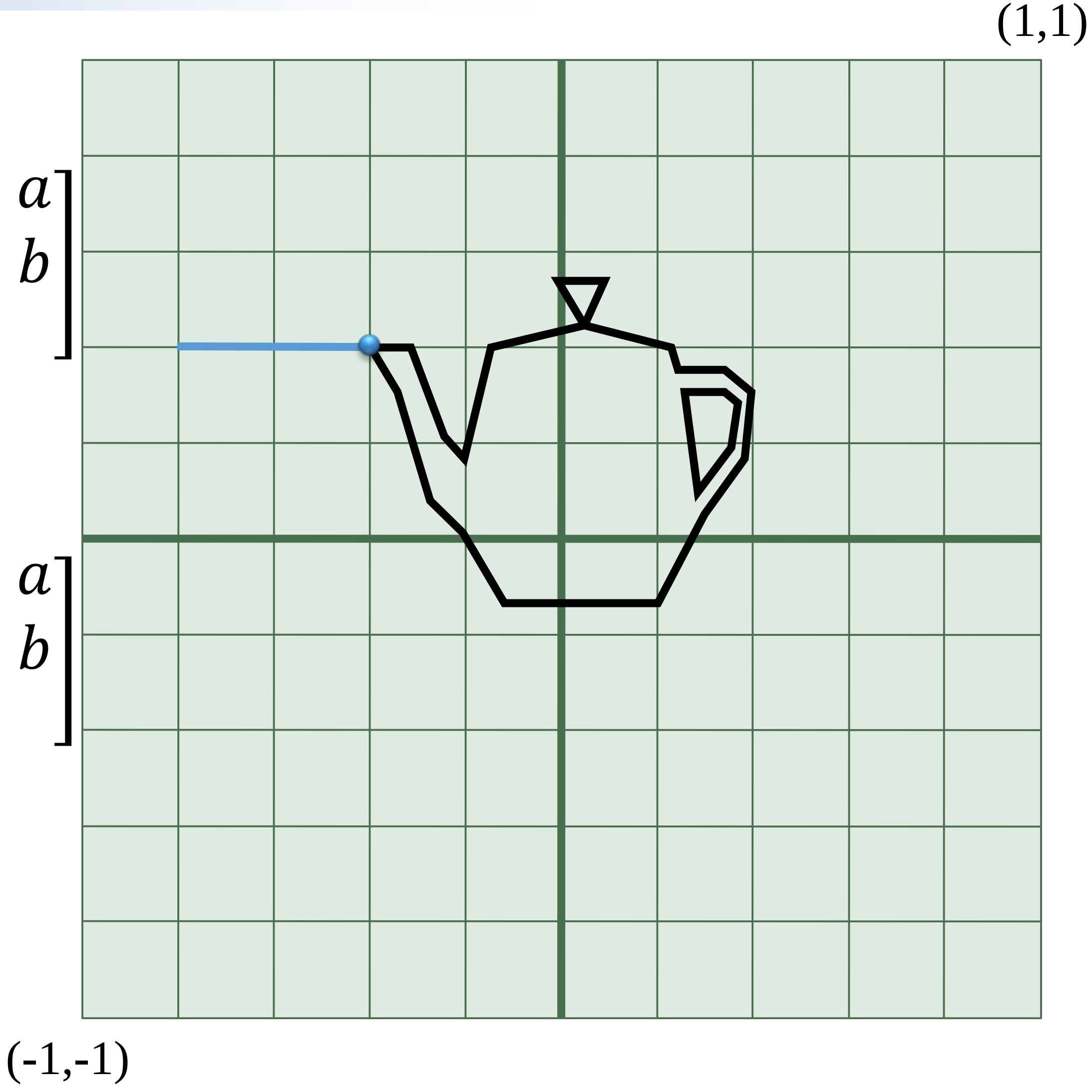
$$\begin{bmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{bmatrix}\left(\begin{bmatrix} s & & \\ & s & \\ & & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} sx + a \\ sy + b \\ 1 \end{bmatrix}$$

- Refactored

$$\left(\begin{bmatrix} 1 & & a \\ & 1 & b \\ & & 1 \end{bmatrix}\begin{bmatrix} s & & \\ & s & \\ & & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} sx + a \\ sy + b \\ 1 \end{bmatrix}$$

- Premultiplied

$$\left(\begin{bmatrix} s & & a \\ & s & b \\ & & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} sx + a \\ sy + b \\ 1 \end{bmatrix}$$



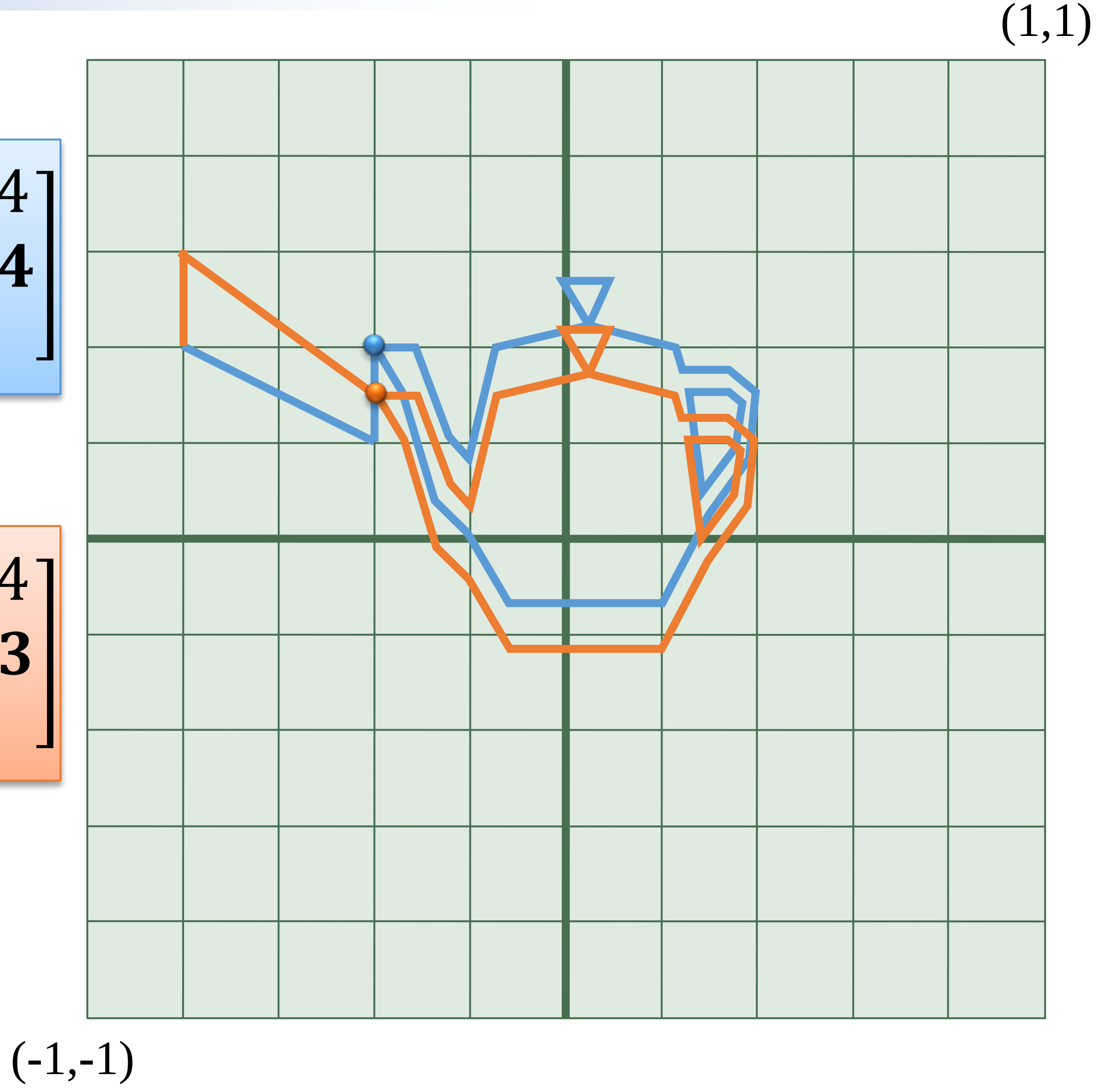
Transformation Composition

- Scale then translate

$$\begin{bmatrix} 1 & & 0 \\ & 1 & 0.2 \\ & & 1 \end{bmatrix}\begin{bmatrix} 0.5 & & \\ & 0.5 & \\ & & 1 \end{bmatrix}\begin{bmatrix} -0.8 \\ 0.4 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.4 \\ \mathbf{0.4} \\ 1 \end{bmatrix}$$

- Translate then scale

$$\begin{bmatrix} 0.5 & & \\ & 0.5 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & 0 \\ & 1 & 0.2 \\ & & 1 \end{bmatrix}\begin{bmatrix} -0.8 \\ 0.4 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.4 \\ \mathbf{0.3} \\ 1 \end{bmatrix}$$



What Have We Learned?

- Shapes are polygonal, and we transform them by transforming their vertex positions
- We represent 2-D positions homogeneously as 3-element column vectors $\begin{bmatrix} x \\ y \\ \mathbf{1} \end{bmatrix}$
- 2-D translation and scale operations are represented by 3x3 homogenous transformation matrices (multiplied on the left)
- **Order matters**